

# JI-DISTRIBUTIVE DUALY QUASI-DE MORGAN LINEAR SEMI-HEYTING ALGEBRAS

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*Dedicated to Professor P.N. Shivakumar  
A Great Humanitarian who changed the course of my life*

**ABSTRACT.** The main purpose of this paper is to axiomatize the join of the variety **DPCSHC** of dually pseudocomplemented semi-Heyting algebras generated by chains and the variety generated by **D<sub>2</sub>**, the De Morgan expansion of the four element Boolean Heyting algebra.

Toward this end, we first introduce the variety **DQDLNSH** of dually quasi-De Morgan linear semi-Heyting algebras defined by the linearity axiom and the variety **JIDSH** of JI-distributive dually quasi-De Morgan semi-Heyting algebras, and present some properties thereof. We then give an explicit description of simple (= subdirectly irreducible) algebras in the variety **JIDLNSH<sub>1</sub>** of JI-distributive dually quasi-De Morgan linear semi-Heyting algebras of level 1 by applying the results of [24] and [25], from which we deduce our main theorem that says that **JIDLNSH<sub>1</sub>** is the join of the variety **DPCSHC** and the variety **V(D<sub>2</sub>)**, solving the problem mentioned above.

We give some applications of this theorem. First, it is shown that the lattice of nontrivial subvarieties of **JIDLNSH<sub>1</sub>** is isomorphic to  $(\omega + 1) \times \mathbf{2}$ , where **2** is the 2-element chain. Then we present (small) bases for all of the subvarieties of **JIDLNSH<sub>1</sub>**. Finally, we show that all subvarieties of **DPCSHC** have the amalgamation property. The paper concludes with some open problems for further investigation.

## 1. Introduction

Semi-Heyting algebras were introduced by us in [23] as an abstraction of Heyting algebras. They share several important properties with Heyting algebras, such as distributivity and pseudocomplementedness. Interestingly, there are also semi-Heyting algebras with properties which, in some sense, are “diagonally opposite” to those of Heyting algebras. For example, the identity  $0 \rightarrow 1 \approx 0$ , as well as the commutative law  $x \rightarrow y \approx y \rightarrow x$ , hold in some semi-Heyting algebras. Further work on semi-Heyting algebras can

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be found in [1] and [2]. More recently, semi-intuitionistic logic, having semi-Heyting algebras as an equivalent algebraic semantics, is given in [7] and [8].

Quasi-De Morgan algebras form a subvariety of semi-De Morgan algebras which are defined and investigated in [22] as a common abstraction of De Morgan algebras and distributive  $p$ -algebras. There have been several papers (including at least one Ph.D. thesis) published, since the appearance of [22].

In 1985, we initiated investigations (see [19]) into an expansion of Heyting algebras by adding the operation of dual pseudocomplementation, and, in 1987, into another expansion of Heyting algebras by adding the De Morgan (more generally, Ockham) operation (see [21]). Striking similarities in the concepts and results obtained in [19] and [21] led us, in 2011, to a unification and generalization of some of those results. More explicitly, an expansion of semi-Heyting algebras by adding a negation, called dual quasi-De Morgan operation, which is a common generalization of the dual pseudocomplementation and De Morgan (the strong negation) operation, is investigated in [24], in order to settle an old conjecture of ours. Also investigated in [24] are double semi-Heyting algebras which generalize double Heyting algebras. Since then, several papers have appeared in this area. (see, for example, [25], [26] and [27].) More recently, a semi-Heyting-Brouwer logic is presented in [9] so that its equivalent algebraic semantics is precisely (the variety of) double semi-Heyting algebras.

The linearity axiom:  $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$ , and the variety of linear Heyting algebras defined by it, have been extensively studied in the literature, both algebraically and logically (see, for instance, [10], [12], [13] and [14]).

We initiated the study of the linear identity in [24] in the context of the variety **DQDSH** of dually quasi-De Morgan semi-Heyting algebras. (However, we did not name the subvariety defined by the linear identity.) It was shown in [24, Lemma 13.1] that the linearity identity implies that the semi-Heyting reducts of **DQDSH**-algebras are (necessrily) Heyting algebras. The axiomatization of **DPCSHC**, the variety of dually pseudocomplemented semi-Heyting algebras generated by chains and that of **V(D<sub>2</sub>)**, the variety generated by the 4-element De Morgan expansion of Boolean Heyting algebra, were obtained in Theorem 13.2 and Theorem 9.2 (c)] of [24], respectively.

These results naturally led us to the problem of axiomatizing the join of the two varieties just mentioned above. The purpose of this paper is to answer this problem. To this end, we define and investigate a new subvariety of **DQDSH**, called **JIDLNSH<sub>1</sub>**, consisting of JI-distributive dually quasi-De Morgan linear semi-Heyting algebras of level 1. Along the way, we (officially) introduce the variety **DQDLNSH** of dually quasi-De Morgan linear semi-Heyting algebras defined by the linearity axiom, and the variety **JIDSH** of JI-distributive dually quasi-De Morgan semi-Heyting algebras (in which the join distributes over the implication in a restricted way), and

present some of their properties required later in this paper. (These two varieties, we believe, are of interest in their own right and deserve further investigation.) We then give an explicit description of simple (= subdirectly irreducible) algebras in  $\mathbf{JIDLNSH}_1$  by applying the results of [24] and [25]. As a consequence, we prove our main theorem of this paper which says that the variety  $\mathbf{JIDLNSH}_1$  is the join of  $\mathbf{DPCSHC}$  and  $\mathbf{V}(\mathbf{D}_2)$ . Clearly, the axioms defining the variety  $\mathbf{JIDLNSH}_1$  provide an axiomatization for the join of  $\mathbf{DPCSHC}$  and  $\mathbf{V}(\mathbf{D}_2)$ , thus solving the problem mentioned above. Several applications of our main theorem are given, including a description of the lattice of subvarieties of  $\mathbf{JIDLNSH}_1$ , and equational axiomatizations of all subvarieties of  $\mathbf{JIDLNSH}_1$ . Finally, we show that all subvarieties of  $\mathbf{DPCSHC}$  have the amalgamation property.

More explicitly, the paper is organized as follows: In Section 2 we recall definitions, notations and results from [24], [25] and [26] needed in the rest of the paper. Section 3 (officially) introduces and investigates the variety  $\mathbf{DQDLNSH}$  of dually quasi-De Morgan linear semi-Heyting algebras defined by the linearity axiom. In Section 4, we define the variety  $\mathbf{JIDSH}$  of JI-distributive dually quasi-De Morgan semi-Heyting algebras and present some of its properties required later in this paper. In Section 5, we give an explicit description of simple (= subdirectly irreducible) algebras in  $\mathbf{JIDLNSH}_1$  by applying the results of [24] and [25], together with those of Section 3 and Section 4. Our main theorem (Theorem 5.9) is then deduced from that description of simple algebras.

Several applications of our main theorem are given in Section 6 and Section 7. In Section 6, it is shown that the lattice of nontrivial subvarieties of  $\mathbf{JIDLNSH}_1$  is isomorphic to  $(\omega + 1) \times \mathbf{2}$ , where  $\mathbf{2}$  is the 2-element chain. We then present (small) bases for all of the subvarieties of  $\mathbf{JIDLNSH}_1$ . In Section 7, we show that all subvarieties of  $\mathbf{DPCSHC}$  have the amalgamation property. Finally, Section 8 concludes with some open problems for further investigation.

## 2. Preliminaries

In this section we recall some notions and results needed to make this paper self-contained. However, for more basic information, we rely on the textbooks [4], [6] and [18], to which we refer the reader.

The following definition is taken from [23].

An algebra  $\mathbf{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$  is a *semi-Heyting algebra* if  $\langle L, \vee, \wedge, 0, 1 \rangle$  is a bounded lattice and  $\mathbf{L}$  satisfies:

- (SH1)  $x \wedge (x \rightarrow y) \approx x \wedge y$
- (SH2)  $x \wedge (y \rightarrow z) \approx x \wedge ((x \wedge y) \rightarrow (x \wedge z))$
- (SH3)  $x \rightarrow x \approx 1$ .

Let  $\mathbf{L}$  be a semi-Heyting algebra. We denote  $x \rightarrow 0$  by  $x^*$  in  $\mathbf{L}$ .  $\mathbf{L}$  is a *Heyting algebra* if  $\mathbf{L}$  satisfies:

$$(H) \quad (x \wedge y) \rightarrow y \approx 1.$$

$\mathbf{L}$  is a *Boolean semi-Heyting algebra* (**BSH**-algebra) if  $\mathbf{L}$  satisfies:

$$(Bo) \quad x \vee x^* \approx 1.$$

Semi-Heyting algebras are distributive and pseudocomplemented, with  $a^*$  as the pseudocomplement of an element  $a$ . We will use these and other properties (see [23]) of semi-Heyting algebras, frequently without explicit mention, throughout this paper.

The following definition, taken from [24], is central to this paper.

**DEFINITION 2.1.** *An algebra  $\mathbf{L} = \langle L, \vee, \wedge, \rightarrow, ', 0, 1 \rangle$  is a semi-Heyting algebra with a dual quasi-De Morgan operation or dually quasi-De Morgan semi-Heyting algebra (**DQDSH**-algebra, for short) if  $\langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$  is a semi-Heyting algebra, and  $\mathbf{L}$  satisfies:*

- (a)  $0' \approx 1$  and  $1' \approx 0$
- (b)  $(x \wedge y)' \approx x' \vee y'$
- (c)  $(x \vee y)'' \approx x'' \vee y''$
- (d)  $x'' \leq x$ .

Let  $\mathbf{L}$  be a **DQDSH**-algebra.  $\mathbf{L}$  is a *dually pseudocomplemented semi-Heyting algebra* (**DPCSH**-algebra) if  $\mathbf{L}$  satisfies:

$$(e) \quad x \vee x' \approx 1.$$

$\mathbf{L}$  is a *De Morgan semi-Heyting algebra*, or *symmetric semi-Heyting algebra*, or *semi-Heyting algebra with strong negation* (**DMSH**-algebra, for short) if  $\mathbf{L}$  satisfies:

$$(DM) \quad x'' \approx x.$$

$\mathbf{L}$  is a *blended dually quasi-De Morgan semi-Heyting algebra* (**BDQDSH**-algebra) if  $\mathbf{L}$  satisfies:

$$(B) \quad (x \vee x^*)' \approx x' \wedge x^{*'} \quad (\text{Blended } \vee\text{-De Morgan law}).$$

$\mathbf{L}$  is a *strongly blended dually quasi-De Morgan semi-Heyting algebra* (**SBDQDSH**, for short) if  $\mathbf{L}$  satisfies:

$$(SB) \quad (x \vee y^*)' \approx x' \wedge y^{*'} \quad (\text{Strongly Blended } \vee\text{-De Morgan law}).$$

The varieties of **DQDSH**-algebras, **DPCSH**-algebras, as well as **DMSH**-algebras are denoted, respectively, by **DQDSH**, **DPCSH** and **DMSH**. Furthermore, **DQDBSH** denotes the subvariety of **DQDSH** defined by (Bo). Similarly, **BDQDSH** and **SBDQDSH** represent, respectively, the variety of **BDQDSH**-algebras and that of **SBDQDSH**-algebras. It is clear that **DMSH**  $\subseteq$  **SBDQDSH**  $\subseteq$  **BDQDSH**. It is proved in [25] that **DPCSH**  $\subseteq$  **SBDQDSH**.

If the underlying semi-Heyting algebra of a **DQDSH**-algebra is a Heyting algebra, then we replace the part “**SH**” by “**H**” in the names of the varieties that will be considered in the sequel.

Note that **DQDStSH** [**DQDStH**] denotes the **DQDSH**- [**DQDStH**-] algebras satisfying the Stone identity:

(St)  $x^* \vee x^{**} \approx 1$ .

The following lemma will be used, often without explicit reference to it. Most of the items in this lemma were proved in [24] and the others are left to the reader.

**LEMMA 2.2.** *Let  $\mathbf{L} \in \mathbf{DQDSH}$  and let  $x, y, z \in L$ . Then*

- (i)  $1'^* = 1$ , and  $1 \rightarrow x = x$
- (ii)  $x \leq y$  implies  $x' \geq y'$
- (iii)  $(x \wedge y)^{I*} = x'^* \wedge y'^*$
- (iv)  $x'^* \leq x^{*'}$
- (v)  $x''' = x'$
- (vi)  $(x \vee y)' = (x'' \vee y'')'$
- (vii)  $(x \vee y)' = (x'' \vee y)'$
- (viii)  $x \leq (x \vee y) \rightarrow x$
- (ix)  $x \leq x \rightarrow 1$
- (x)  $x \wedge [(x \rightarrow y) \rightarrow z] = x \wedge (y \rightarrow z)$
- (xi)  $x \vee x^+ = 1$
- (xii)  $x \wedge [y \vee (x \rightarrow z)] \approx x \wedge (y \vee z)$
- (xiii)  $x \wedge [(x \vee y) \rightarrow z] = x \wedge z$
- (xiv)  $x \wedge [(x \wedge y) \rightarrow z] = x \wedge (y \rightarrow z)$
- (xv)  $x \wedge [y \rightarrow (x \rightarrow z)] = x \wedge (y \rightarrow z)$
- (xvi)  $x' \vee (x \rightarrow y)' \approx x' \vee y'$
- (xvii)  $x \wedge [y \rightarrow (z \rightarrow x)] = x \wedge (y \rightarrow z)$
- (xvii)  $(x' \vee y)' \leq x$
- (xix)  $x \wedge (x \rightarrow y)'' \leq y$ .

Figure 1 defines the 4-element algebra **D<sub>2</sub>** (as named in [24]) whose lattice reduct is the Boolean lattice having the universe  $\{0, a, b, 1\}$ , with  $b$  as the complement of  $a$ , and  $'$  is defined as follows:  $a' = a$ ,  $b' = b$ ,  $0' = 1$  and  $1' = 0$ , and  $\rightarrow$  is defined in below.

$\rightarrow$	0	1	$a$	$b$
0	1	1	1	1
<b>D<sub>2</sub></b> : 1	0	1	$a$	$b$
$a$	$b$	1	1	$b$
$b$	$a$	1	$a$	1

Figure 1

It is clear that  $\mathbf{D}_2 \in \mathbf{DMH}$ .

**LEMMA 2.3.** *Let  $\mathbf{L}$  be a  $\mathbf{DQDSH}$ -algebra. Let  $a$  be in  $L$  such that  $a' = a$ . Then  $a^{*'} = a^*$ .*

*Proof.*  $1 = 0' = (a \wedge a^*)' = a' \vee a^{*'}$ , implying  $a \vee a^{*'} = 1$ . It then follows that  $a^* \leq a^{*'}$ . Then  $a^{*' } = (a^* \wedge a^{*' })' = a^{*' } \vee a^{*''}$ . Thus,  $a^{*' } \leq a^{*''} \leq a^*$ , yielding that  $a^{*' } = a^*$ .  $\square$

The following definition is from [24].

**DEFINITION 2.4.** *Let  $\mathbf{L} \in \mathbf{DQDSH}$  and  $x \in \mathbf{L}$ . For  $n \in \omega$ , we define  $t_n(x)$  recursively as follows:*

$$\begin{aligned} x^{0(\iota^*)} &= x; \\ x^{(n+1)(\iota^*)} &= (x^{n(\iota^*)})'^*, \text{ for } n \geq 0; \\ t_0(x) &= x, t_{n+1}(x) = t_n(x) \wedge x^{(n+1)(\iota^*)}, \text{ for } n \geq 0. \end{aligned}$$

For  $n \in \omega$ , the  $n$ th level (or level  $n$ ) subvariety  $\mathbf{DQDSH}_n$  of  $\mathbf{DQDSH}$  is defined by the identity:

$$t_n(x) \approx t_{n+1}(x).$$

For  $n \in \omega$ , we let  $\mathbf{BDQDSH}_n := \mathbf{BDQDSH} \cap \mathbf{DQDSH}_n$ . In a similar fashion,  $\mathbf{DMSH}_n$ , etc. are defined.

Now, we recall some known results that are needed later in this paper.

**PROPOSITION 2.5.** [25, Theorem 2.5] *For  $n \in \omega$ ,  $\mathbf{DQDStSH}_n$  is defined by the identity:  $(x \wedge x'^*)^{n(\iota^*)} \approx (x \wedge x'^*)^{(n+1)(\iota^*)}$ , modulo  $\mathbf{DQDStSH}$ .*

*In particular, the variety  $\mathbf{DQDStSH}_1$  is defined by the identity:*

$$(L1) \ x \wedge x'^* \approx (x \wedge x'^*)'^*,$$

*relative to  $\mathbf{DQDStSH}$ .*

**PROPOSITION 2.6.** [25, Theorem 3.4]  $\mathbf{DQDStSH}_1 = \mathbf{SBDQDStSH}_1$ .

**THEOREM 2.7.** ([25, Corollary 4.1]) Let  $\mathbf{L} \in \mathbf{BDQDSH}_1$  with  $|L| \geq 2$ . Then the following conditions are equivalent:

- (1)  $\mathbf{L}$  is simple
- (2)  $\mathbf{L}$  is subdirectly irreducible
- (SC) For every  $x \in L$ , if  $x \neq 1$ , then  $x \wedge x'^* = 0$ .

**THEOREM 2.8.** ([24, Corollary 8.2(a)]) For  $n \in \omega$ ,  $\mathbf{BDQDSH}_n$  is a discriminator variety.

### 3. THE VARIETY OF DUALLY QUASI-DE MORGAN LINEAR SEMI-HEYTING ALGEBRAS

As mentioned in the introduction, we initiated in [24] the study of the linearity axiom:  $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$  in the context of dually quasi-De Morgan semi-Heyting algebras; however, we did not isolate the subvariety defined by the linear identity by giving it a name. We will do it now.

**DEFINITION 3.1.** *A semi-Heyting algebra  $\mathbf{L}$  is linear if it satisfies:*

(LN)  $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$  (Linearity axiom, or Linearity identity).

**DQDLNSH** [**DQDLNH**] denotes the subvariety of **DQDSH** [**DQDH**] defined by (LN).

The variety **DQDLNSH** will be the focus of this section.

In what follows, **DQDLNSH** is abbreviated as **DQDLN**. **DQDLNH** denotes the variety of dually quasi-De Morgan linear Heyting algebras. We recall the following known result (stated in the current terminology), which is needed later in this paper.

**PROPOSITION 3.2.** [24, Lemma 12.1(f)] *Let  $\mathbf{L}$  be a linear semi-Heyting algebra. Then*

- (a)  $\mathbf{L} \models (\text{H})$ .
- (b)  $\mathbf{DQDLN} \subset \mathbf{DQDLNH}$ .

**LEMMA 3.3.** *Let  $\mathbf{L} \in \mathbf{DQDLN}$  and let  $x, y \in L$ . Then*

- (a)  $(x \rightarrow y) \vee (y \rightarrow x)'' = 1$
- (b)  $x \leq y \vee (y \rightarrow x)''$ .

*Proof.* For (a),

$$\begin{aligned} (x \rightarrow y) \vee (y \rightarrow x)'' &\geq (x \rightarrow y)'' \vee (y \rightarrow x)'' \\ &= [(x \rightarrow y) \vee (y \rightarrow x)]'' \\ &= 1 \quad \text{by (LN).} \end{aligned}$$

To prove (b),

$$\begin{aligned} x \wedge [y \vee (y \rightarrow x)''] &= x \wedge [(x \rightarrow y) \vee (y \rightarrow x)''] \quad \text{by Lemma 2.2(xii)} \\ &= x \quad \text{by (a).} \end{aligned}$$

□

Recall from [24] that an algebra in **DQDSH** is a **DQDSH**-chain if its lattice reduct is a chain. **DQDSHC** denotes the variety generated by the **DQDSH**-chains. For  $\mathbf{L}$  a **DPCSH**-chain, we note that the dual pseudo-complement  $'$  satisfies:  $a' = 1$ , if  $a \neq 1$ , and  $a' = 0$ , if  $a = 1$ . **DPCSHC** denotes the subvariety of **DPCSH** generated by all **DPCSH**-chains. The following lemma, which is easy to prove, provides a large class of algebras in **DQDLN**.

**LEMMA 3.4.**  $\mathbf{DQDHC} \subset \mathbf{DQDLN}$ .

*In particular,  $\mathbf{DPCHC} \subset \mathbf{DQDLN}$ .*

It is crucial to point out that the linear identity also holds in such algebras as  $\mathbf{D}_2$ , that are not **DQDSH**-chains. Since  $\mathbf{D}_2$  is simple, it follows that the variety **DQDLN** is not generated by the chain-based algebras. (We note, therefore, that the term “linear“ loses its original meaning, as used in the context of Heyting algebras. Nevertheless, we have retained the term for the sake of continuity.)

**LEMMA 3.5.** *Let  $\mathbf{L}$  be a linear semi-Heyting algebra. Then  $L \models (\text{St})$ .*

*Proof.* Observe from Proposition 3.2 that  $\mathbf{L}$  is actually a Heyting algebra. Therefore, applying the results of [14], it is easy to see that  $\mathbf{L}$  satisfies (St).  $\square$

The following theorem is useful later.

**THEOREM 3.6.**  $\mathbf{DQDHC} \subset \mathbf{DQDLN} \subset \mathbf{DQDStH}$ .

*Proof.* The first half is immediate from Lemma 3.4, while the second half follows from Lemma 3.5.  $\square$

It should be noted that  $\mathbf{DQDLN} \not\subseteq \mathbf{DQDSH}_1$ .

**EXAMPLE 3.7.** *Let us consider the algebra **SIX** whose lattice reduct,  $\rightarrow$  and  $'$  are given below:*

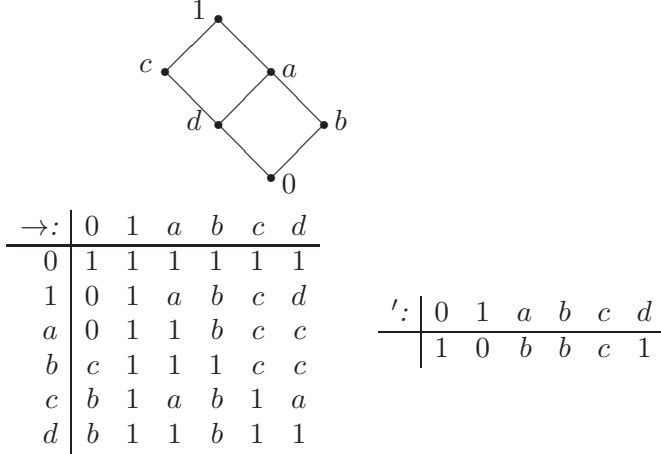


Figure 2

Observe that the algebra **SIX** is in **DQDLN** but doesn't satisfy (L1) (at  $a$ ), and hence is not in **DQDSH**<sub>1</sub>. This observation leads us naturally to the next definition.

**DEFINITION 3.8.**  $\mathbf{DQDLNSH}_1 := \mathbf{DQDLN} \cap \mathbf{DQDSH}_1$ .

In particular,  $\mathbf{DQDLNH}_1 := \mathbf{DQDLN} \cap \mathbf{DQDH}_1$ .

$\mathbf{DQDLNSH}_1$  will be abbreviated to  $\mathbf{DQDLN}_1$ .

**PROPOSITION 3.9.**  $\mathbf{DQDLN}_1 \subset \mathbf{SBDQDStSH}_1 \subset \mathbf{BDQDStSH}_1$ .



*Proof.* The first inclusion is immediate from Theorem 3.6 and Proposition 2.6, and it is proper since the algebra  $\mathbf{SIX} \in \mathbf{DQDLN} \setminus \mathbf{DQDLN}_1$ . The second inclusion was already noted in [24].  $\square$

The above proposition allows us to apply Theorem 2.7 later in Section 5.

#### 4. THE VARIETY **JIDSH** OF **JI-DISTRIBUTIVE DQDSH**-ALGEBRAS

The identity  $x \vee (y \rightarrow z) \approx (x \vee y) \rightarrow (x \vee z)$  was considered in [26, Corollary 3.55]. We will call this identity as *strong JI-distributive identity*. We now introduce a slight weakening of this identity, called *JI-distributive identity*, which, along with linearity identity, plays a crucial role in the rest of this paper.

**DEFINITION 4.1.** *The subvariety **JIDSH** of **DQDSH** is defined by:*

(JID)  $x' \vee (y \rightarrow z) \approx (x' \vee y) \rightarrow (x' \vee z)$  (**J**oin **D**istributes over **I**mplication).

In what follows we will abbreviate **JIDSH** by **JID**. However, if the underlying semi-Heyting algebra is a Heyting algebra, we will use **JIDH** for **JIDSH**.

**LEMMA 4.2.** *Let  $\mathbf{L} \in \mathbf{JID}$ , and  $x, y \in \mathbf{L}$ . Then  $\mathbf{L}$  satisfies:*

- (1)  $x' \rightarrow (x' \vee y) \approx x' \vee (x' \rightarrow y)$
- (2)  $x' \rightarrow (x' \vee y) \approx x' \vee (0 \rightarrow y)$
- (3)  $x' \vee (x' \rightarrow y) \approx x' \vee (0 \rightarrow y)$
- (4) *If  $\mathbf{L} \models 0 \rightarrow y \approx 1$ , then  $x' \vee (x' \rightarrow y) \approx 1$*
- (5)  $(x' \vee y) \rightarrow x' \approx x' \vee y^*$
- (6)  $(x' \vee y) \rightarrow x' \approx x' \vee (y \rightarrow x')$  (*Exchage Property*)
- (7)  $x' \vee (y \rightarrow x') \approx x' \vee y^*$
- (8) *If  $\mathbf{L} \models y \leq x \rightarrow y$ , then  $\mathbf{L} \models x \rightarrow y' \approx x^* \vee y'$ .*

*Proof.* Let  $x, y \in \mathbf{L}$ . Then

$$\begin{aligned} x' \rightarrow (x' \vee y) &= (x' \vee x') \rightarrow (x' \vee y) \\ &= x' \vee (x' \rightarrow y) \quad \text{by the axiom (JID),} \end{aligned}$$

which proves (1). To prove (2), we have

$$\begin{aligned} x' \vee (0 \rightarrow y) &= (x' \vee 0) \rightarrow (x' \vee y) \quad \text{by the axiom (JID),} \\ &= x' \rightarrow (x' \vee y). \end{aligned}$$

(3) is immediate from (1) and (2), and (4) follows from (3).

For (5),

$$\begin{aligned} (x' \vee y) \rightarrow x' &= (x' \vee y) \rightarrow (x' \vee 0) \\ &= x' \vee (y \rightarrow 0) \quad \text{by (JID)} \\ &= x' \vee y^*. \end{aligned}$$

To prove (6),

$$\begin{aligned} (x' \vee y) \rightarrow x' &= (x' \vee y) \rightarrow (x' \vee x') \\ &= x' \vee (y \rightarrow x') \quad \text{by (JID)}. \end{aligned}$$

(7) is immediate from (5) and (6); and (8) follows from (7) since  $y' \leq x \rightarrow y'$ , as  $\mathbf{L} \models (\text{H})$ . □

**LEMMA 4.3.** *Let  $\mathbf{L} \in \mathbf{JID}$  satisfy the condition:*

(SC) *For every  $x \in L$ , if  $x \neq 1$  then  $x \wedge x'^* = 0$ .*

*Let  $a \in \mathbf{L}$  such that  $a \neq a'$ . Then*

- (a)  $a = 1$  or  $a' = 1$
- (b)  $a \vee a' = 1$ .

*Proof.* Suppose  $a \neq 1$  and  $a' \neq 1$ . Then  $a' \wedge a''^* = 0$  by (SC). So,  $a' \wedge (a''^* \vee a'') = a' \wedge a''$ . But we know from Lemma 4.2 (3) that  $a'' \vee a''^* = 1$ . Hence  $a' = a' \wedge a''$ , implying  $a' \leq a$ . Also,  $a \wedge a'^* = 0$  by (SC). So,  $a \wedge (a' \vee a'^*) = a'$ , which, by Lemma 4.2 (3), implies that  $a = a'$ , which is a contradiction, proving (a). Now, (b) is immediate from (a). □

**LEMMA 4.4.** *Let  $\mathbf{L} \in \mathbf{JID}$  such that  $\mathbf{L} \models 0 \rightarrow x \approx 1$ . Then*

$$\mathbf{L} \models x^* \leq x \rightarrow y.$$

*Proof.* Let  $x, y \in \mathbf{L}$ .

$$\begin{aligned} x^* \wedge (x \rightarrow y) &= x^* \wedge [(x^* \wedge x) \rightarrow (x^* \wedge y)] \\ &= x^* \wedge (0 \rightarrow x) \\ &= x^* \quad \text{by } 0 \rightarrow x \approx 1. \end{aligned}$$
□

**LEMMA 4.5.** *Let  $\mathbf{L} \in \mathbf{DQDStSH}$  and  $x, y \in \mathbf{L}$  such that  $x^{**} = x$ . Then  $(x^* \wedge y')^* = x \vee y'^*$ .*

*Proof.*

$$\begin{aligned} (x^* \wedge y')^* &= x^{**} \vee y'^* \quad \text{by a well known property of Stone algebras,} \\ &\quad \text{as } \mathbf{L} \models (\text{St}) \\ &= x \vee y'^* \quad \text{since } a^{**} = a. \end{aligned}$$
□

Recall that **JIDH** is the subvariety of **JID** defined by (H). Note, therefore, that  $\mathbf{JIDH} \models 0 \rightarrow x \approx 1$ ,  $\mathbf{JIDH} \models x \leq y \rightarrow x$ .

**LEMMA 4.6.** *Let  $\mathbf{L} \in \mathbf{JIDH}$ , and  $a, x \in \mathbf{L}$  with  $a' = a$ .*

- (1)  $a \vee (a \rightarrow x) = 1$
- (2)  $a \vee [(a \rightarrow x) \rightarrow y] = a \vee y$
- (3)  $(a \rightarrow x) \rightarrow x = a \vee x$
- (4)  $(a \vee x) \wedge (a \rightarrow x) = x$

- (5)  $a \vee (a \rightarrow x)' = a \vee x'$
- (6)  $(a \vee x') \wedge [a \rightarrow (a \rightarrow x)'] = (a \rightarrow x)'$
- (7)  $(a \vee x') \wedge [(a \rightarrow x)' \vee a^*] = (a \rightarrow x)'$
- (8)  $a^* \geq (a \rightarrow x)'$
- (9)  $(a \rightarrow x)' = a^* \wedge x'$
- (10)  $(a \vee x)' \rightarrow a' = 1$
- (11)  $a \vee (a \vee x)'^* = 1$ .

*Proof.* We have

(1)

$$\begin{aligned} a \vee (a \rightarrow x) &= a' \vee (a' \rightarrow x) \quad \text{since } a' = a \\ &= 1 \quad \text{by Lemma 4.2 (4)} \end{aligned}$$

(2)

$$\begin{aligned} a \vee [(a \rightarrow x) \rightarrow y] &= [a \vee (a \rightarrow x)] \rightarrow (a \vee y) \quad \text{by (JID)} \\ &= 1 \rightarrow (a \vee y) \quad \text{by (1)} \\ &= a \vee y. \end{aligned}$$

(3)

$$\begin{aligned} a \vee x &= a \vee [(a \rightarrow x) \rightarrow x] \quad \text{by (2)} \\ &= (a \rightarrow x) \rightarrow x \quad \text{by Lemma 2.2 (x)}. \end{aligned}$$

(4)

$$\begin{aligned} (a \vee x) \wedge (a \rightarrow x) &= [(a \rightarrow x) \rightarrow x] \wedge (a \rightarrow x) \quad \text{by (3)} \\ &= (a \rightarrow x) \wedge x \quad \text{by (SH1)} \\ &= x \quad \text{since } x \leq a \rightarrow x, \text{ as } \mathbf{L} \models (\mathbf{H}). \end{aligned}$$

(5)

$$\begin{aligned} a \vee (a \rightarrow x)' &= a' \vee (a' \rightarrow x)' \quad \text{since } a' = a \\ &= [a \wedge (a \rightarrow x)]' \\ &= (a \wedge x)' \\ &= a \vee x' \quad \text{since } a' = a. \end{aligned}$$

(6)

$$\begin{aligned} (a \rightarrow x)' &= [a \vee (a \rightarrow x)'] \wedge [a \rightarrow (a \rightarrow x)'] \quad \text{by (4)} \\ &= (a \vee x)' \wedge [a \rightarrow (a \rightarrow x)'] \quad \text{by (5)}. \end{aligned}$$

(7)

$$\begin{aligned} (a \rightarrow x)' &= (a \vee x') \wedge [a \rightarrow (a \rightarrow x)'] \quad \text{by (6)} \\ &= (a \vee x') \wedge [a^* \vee (a \rightarrow x)'] \quad \text{by Lemma 4.2 (8)}. \end{aligned}$$

(8) Since  $a^* \wedge (a \rightarrow x) = a^*$  by Lemma 4.4, we have

$$a^{*'} \vee (a \rightarrow x)' = a^{*'},$$

which implies  $a^* \vee (a \rightarrow x)' = a^*$  by Lemma 2.3, since  $a' = a$ .

(9)

$$\begin{aligned} (a \rightarrow x)' &= (a \vee x') \wedge [a^* \vee (a \rightarrow x)'] && \text{by (7)} \\ &= (a \vee x') \wedge a^* && \text{by (8)} \\ &= x' \wedge a^*. \end{aligned}$$

(10)

$$\begin{aligned} (a \vee x)' \rightarrow a' &= (a \vee x)' \rightarrow [a' \vee (a \vee x)'] \\ &= (a \vee x)' \rightarrow [a \vee (a \vee x)'] && \text{since } a' = a \\ &= 1 && \text{since } \mathbf{L} \models (\mathbf{H}). \end{aligned}$$

(11)

$$\begin{aligned} a \vee (a \vee x)^{ '* } &= (a \vee x)^{ '* } \vee a' && \text{since } a' = a \\ &= (a \vee x)' \rightarrow a' && \text{by Lemma 4.2 (8)} \\ &= 1 && \text{by (10)}. \end{aligned}$$

□

Recall that **DPCSHC** = **DPCHC**.

**THEOREM 4.7.** **DPCSHC**  $\models$  (JID).

*Proof.* Suppose that the identity (JID) fails in **DPCSHC**. Then there is a **DPCSH**-chain **A** and elements  $a, b, c \in A$  such that

$$(1) \quad a' \vee (b \rightarrow c) \neq (a' \vee b) \rightarrow (a' \vee c).$$

First we claim that  $a = 1$  or  $a' = 1$ . For,  $a' \leq a$  or  $a \leq a'$  (since **A** is a chain), which implies that

$$(2) \quad a \vee a' \leq a \text{ or } a \vee a' \leq a'.$$

Since **A** is dually pseudocomplemented, we have  $a \vee a' = 1$ . Hence it follows from (2) that  $a = 1$  or  $a' = 1$ , proving the claim. Suppose  $a' = 1$ . Then the statement (1) reduces to  $1 \neq 1$ , which is a contradiction. Next, suppose that  $a = 1$ . Then the statement (1) simplifies to  $b \rightarrow c \neq b \rightarrow c$ , which is a contradiction. Hence the proof is complete. □

**PROPOSITION 4.8.** Let **L**  $\in$  **JID** satisfy:  $0 \rightarrow y \approx 1$ . Then **L** satisfies:

- (1)  $x' \rightarrow (x' \vee y) \approx 1$
- (2)  $x' \vee (x' \rightarrow y) \approx 1$
- (3)  $x' \vee [(x \vee y)' \rightarrow z] \approx 1$
- (4)  $x \vee (x'' \rightarrow y) \approx 1$
- (5)  $x \vee [(x' \vee y)' \rightarrow z] \approx 1$ .

*Proof.* Let  $x, y \in L$ . Then

For (1),

$$\begin{aligned} x' \rightarrow (x' \vee y) &= x' \vee (0 \rightarrow y) \quad \text{by Lemma 4.2 (2)} \\ &= 1 \quad \text{by the hypothesis: } 0 \rightarrow y \approx 1. \end{aligned}$$

To prove (2),

$$\begin{aligned} x' \vee (x' \rightarrow y) &= x' \rightarrow (x' \vee y) \quad \text{by Lemma 4.2(1)} \\ &= 1 \quad \text{by (1).} \end{aligned}$$

Next, for (3),

$$\begin{aligned} x' \vee [(x \vee y)' \rightarrow z] &= [\{x' \vee (x \vee y)'\} \rightarrow (x' \vee z)] \quad \text{by (JID)} \\ &= x' \rightarrow (x' \vee z) \quad \text{by Lemma 2.2(ii).} \\ &= 1 \quad \text{by (1).} \end{aligned}$$

To prove (4),

$$\begin{aligned} x \vee (x'' \rightarrow y) &= x \vee x'' \vee (x'' \rightarrow y) \\ &= 1 \quad \text{by (2).} \end{aligned}$$

For (5),

$$\begin{aligned} x \vee [(x' \vee y)' \rightarrow z] &= x \vee (x' \vee y)' \vee [(x' \vee y)' \rightarrow z] \quad \text{by Lemma 2.2(xvii)} \\ &= 1 \quad \text{by (2).} \end{aligned}$$

□

## 5. THE SIMPLICITY IN THE VARIETY $\mathbf{JIDLN}_1$

In this section we present an explicit description of subdirectly irreducible (= simple) algebras in the variety  $\mathbf{JIDLNSH}_1$  of JI-distributive dually quasi-De Morgan linear semi-Heyting algebras of level 1 and then derive the main result of this paper.

**DEFINITION 5.1.** Let  $\mathbf{JIDLNSH} := \mathbf{JIDSH} \cap \mathbf{DQDLNSH}$ .

We abbreviate  $\mathbf{JIDLNSH}$  to  $\mathbf{JIDLN}$ , and  $\mathbf{JIDLNH}$  denotes  $\mathbf{JIDLNSH}$  when the underlying semi-Heyting algebra is actually a Heyting algebra. Note that the algebra **SIX** given in the last section is, in fact, in  $\mathbf{JIDLN}$ . Hence  $\mathbf{JIDLN} \not\subseteq \mathbf{DQDSH}_1$ . However, it turns out that  $\mathbf{JIDLN} \subseteq \mathbf{DQDSH}_2$ —a result that will appear in a future publication.

Now, we are ready to define the variety of JI-distributive linear quasi-De Morgan semi-Heyting algebras of level 1 that would help us solve the problem raised in the Introduction.

**DEFINITION 5.2.** Let  $\mathbf{JIDLN}_1 := \mathbf{JID} \cap \mathbf{LN}_1$ .

In the rest of the paper we focus our attention on the variety  $\mathbf{JIDLN}_1$ .

**THEOREM 5.3.**  $\mathbf{DPCSHC} \vee \mathbf{V(D}_2) \subseteq \mathbf{JIDLN}_1$ .

*Proof.* We know from Theorem 4.7 that **DPCSHC** satisfies (JID), and from Lemma 3.4 that **DPCSHC** satisfies (LN), while it is clear that **DPCSHC** also satisfies (L1). We know from [26] that **D<sub>2</sub>** satisfies the strong (JID), so **D<sub>2</sub>**  $\models$  (JID). It is routine to verify that **D<sub>2</sub>** also satisfies (LN) and (L1). Hence the theorem follows.  $\square$

We now focus on proving the reverse inclusion of the above theorem.

**Unless otherwise stated, in the rest of this section we assume that  $\mathbf{L} \in \mathbf{JIDLN}_1$  and satisfies the simplicity condition:**

(SC) For every  $x \in L$ , if  $x \neq 1$  then  $x \wedge x'^* = 0$ .

**LEMMA 5.4.** *Let  $a, b \in L$  such that  $a' = a$ . Then*

- (i)  $a \rightarrow b = 1$  or  $a \rightarrow b = a^*$
- (ii)  $a \leq b$  or  $a \wedge b = 0$
- (iii)  $a \rightarrow b = b$  or  $b \leq a$ .

*Proof.* Suppose  $a \rightarrow b \neq 1$ . Then  $(a \rightarrow b) \wedge (a \rightarrow b)^{*'} = 0$  by (SC). Hence,

$$(3) \quad [(a \rightarrow b) \vee a^*] \wedge [(a \rightarrow b)^{*'} \vee a^*] = a^*.$$

Since  $\mathbf{L} \models$  (LN), we get  $\mathbf{L} \models$  (H) by Proposition 3.2, and hence we have  $\mathbf{L} \models 0 \rightarrow x \approx 1$ . Therefore, by Lemma (4.4), the equation (3) simplifies to  $(a \rightarrow b) \wedge [(a \rightarrow b)^{*'} \vee a^*] = a^*$ . From this, in view of Lemma 4.6 (9), we get

$$(4) \quad (a \rightarrow b) \wedge [(a^* \wedge b')^* \vee a^*] = a^*.$$

Observe that  $a^{**} = a$  by Lemma 4.6 (3). Hence, by Lemma 4.5, the equation (4) simplifies to  $(a \rightarrow b) \wedge (a \vee a^* \vee b'^*) = a^*$ . Since  $a \vee a^* = 1$  by Lemma 4.6 (1), we have  $a^* = a \rightarrow b$ , proving (i). From (i) we get  $a \wedge (a \rightarrow b) = a$  or  $a \wedge (a \rightarrow b) = 0$ , from which (ii) follows.

Suppose  $a \vee b \neq 1$ . Then  $(a \vee b) \wedge (a \vee b)^{*'} = 0$  by (SC), from which we get  $(a \vee b) \wedge [a \vee (a \vee b)^{*'}] = a$ , which, in view of Lemma 4.6 (11), simplifies to  $a \vee b = a$ . Thus we have proved

$$(5) \quad a \vee b = 1 \text{ or } b \leq a.$$

From (5) we get  $(a \vee b) \wedge (a \rightarrow b) = a \rightarrow b$  or  $b \leq a$ , which, by Lemma 4.6 (4), yields  $a \rightarrow b = b$  or  $b \leq a$ , proving (iii).  $\square$

**THEOREM 5.5.** *Let  $\mathbf{L} \in \mathbf{JIDLN}_1$  and  $a \in L$  such that  $a' = a$ . Then  $\mathbf{L} \cong \mathbf{D}_2$ .*

*Proof.* Let  $b \in L$  be arbitrary such that  $0 < b < 1$  and  $a \neq b$ . We claim that  $b = a^*$ . For, suppose  $b \neq a^*$ . Then, from (i) and (iii) of Lemma 5.4 we get

$$(6) \quad a \rightarrow b = 1 \text{ or } b \leq a.$$

Next, observe that (6) and Lemma 5.4 (iii) imply that  $b = 1$  or  $b \leq a$ . As  $b \neq 1$  by hypothesis, we conclude that  $b \leq a$ . Also, we know  $a \leq b$  or  $a \wedge b = 0$  from Lemma 5.4 (ii). Hence it follows that  $a = b$  or  $b = 0$ , which contradicts the assumption, proving the claim. Hence, we have that  $L = \{0, a, a^*, 1\}$ .

Since  $a \vee a^* = 1$  by Lemma 4.6 (1),  $\mathbf{L}$  has a Boolean semi-Heyting reduct. Furthermore, we know that  $\mathbf{L} \models (\text{H})$ , implying that  $\mathbf{L} \models 0 \rightarrow 1 \approx 1$ . Therefore, we conclude from [24, Theorem 9.5] that  $\mathbf{L} \cong \mathbf{D}_2$ .  $\square$

**LEMMA 5.6.** *Let  $x, y \in L$ . Then*

- (1)  $x \neq 1 \Rightarrow x \wedge (y \vee x'^*) = x \wedge y$
- (2)  $x \vee y \neq 1 \Rightarrow x \leq y'$ .

*Proof.* Let  $x \neq 1$ , then by (SC) we have  $x \wedge (y \vee x^*) = (x \wedge y) \vee (x \wedge x'^*) = x \wedge y$ , implying (1). Let  $x \vee y \neq 1$ ; then  $(x \vee y) \wedge [y' \vee (x \vee y)^{*}] = (x \vee y) \wedge y'$  by (1), from which it is clear that  $x \wedge [y' \vee (x \vee y)^{*}] = x \wedge y'$ , which, in view of Lemma 4.8(3), implies  $x \wedge 1 = x \wedge y'$ , proving (2).  $\square$

**LEMMA 5.7.** *Let  $a, b \in L$  such that  $a' \neq a$ ,  $a \not\leq b$ . Then  $(a \rightarrow b)'' = 0$ .*

*Proof.* From Lemma 4.3 (a) we have  $a = 1$  or  $a' = 1'$ . Since, by hypothesis,  $b \not\leq a$ , we see that  $a \neq 1$ . Hence

$$(7) \quad a' = 1.$$

Next, we claim that  $a \not\leq (a \rightarrow b)''$ . For, suppose  $a \leq (a \rightarrow b)''$ ; then  $a = a \wedge (a \rightarrow b)'' \leq b$  by Lemma 2.2 (xix), implying  $a \leq b$ , which is a contradiction to the hypothesis  $a \not\leq b$ . Hence  $a \not\leq (a \rightarrow b)''$ . Then  $a \vee (a \rightarrow b)' = 1$  by (the contrapositive of) Lemma 5.6 (2), yielding  $[a \vee (a \rightarrow b)']' = 0$ , which implies  $[a'' \vee (a \rightarrow b)''']' = 0$ . This, by (7), simplifies to  $(a \rightarrow b)'' = 0$ , proving the lemma.  $\square$

We are now ready to give an explicit description of simple algebras in the variety  $\mathbf{JIDLN}_1$ . Observe, by Proposition 3.9, that  $\mathbf{JIDLN}_1 \subseteq \mathbf{DQDLN}_1 \subseteq \mathbf{SBDQDSH}_1 \subseteq \mathbf{BDQDSH}_1$ . Hence, we obtain such a description as an application of Theorem 2.7 (proved in [25]).

**PROPOSITION 5.8.** *Let  $\mathbf{L} \in \mathbf{JIDLN}_1$ . Then the following are equivalent:*

- (1)  $\mathbf{L}$  is subdirectly irreducible
- (2)  $\mathbf{L}$  is simple
- (3)  $\mathbf{L}$  satisfies (SC).
- (4) Either  $\mathbf{L} \cong \mathbf{D}_2$  or  $\mathbf{L}$  is a **DPCH**-chain.

*Proof.* (1)  $\Leftrightarrow$  (2) and (2)  $\Leftrightarrow$  (3) follow from Theorem 2.7, and it is routine to verify that (4)  $\Rightarrow$  (3). So we only need to prove (3)  $\Rightarrow$  (4). Suppose (3) is true in  $\mathbf{L}$ . We consider the two cases: First, suppose there is an  $a \in L$  such that  $a' = a$ . Then, by Theorem 5.5,  $\mathbf{L} \cong \mathbf{D}_2$ . Next, suppose  $\mathbf{L}$  satisfies:

$$(8) \quad \text{For every } x \in L, \ x' \neq x.$$

Then, in view of (8), we get from Lemma 4.3 that  $\mathbf{L}$  is dually pseudocomplemented, and so,  $\mathbf{L} \in \mathbf{DPCSH}_1$ . Next, let  $a, b \in L$  be arbitrary such that  $a \neq b$ . Assume, if possible,  $a \not\leq b$  and  $b \not\leq a$ . Then, from Lemma 3.3(b) we have that  $b \leq a \vee (a \rightarrow b)''$ , which, in view of Lemma 5.7, implies  $b \leq a$ , which contradicts the assumption, thus proving that  $a \leq b$  or  $b \leq a$ . Hence

we conclude that the lattice reduct of  $\mathbf{L}$  is a chain. Since  $\mathbf{L} \models (\text{LN})$ ,  $\mathbf{L}$  is a Heyting algebra. Therefore, the semi-Heyting reduct of  $\mathbf{L}$  is a Heyting chain, leading to the conclusion that  $\mathbf{L}$  is a **DPCH**-chain. Thus we have proved that (3)  $\Rightarrow$  (4), which completes the proof.  $\square$

We have now arrived at the main result of this paper.

**THEOREM 5.9.**  $\mathbf{JIDLN}_1 = \mathbf{DPCSHC} \vee \mathbf{V}(\mathbf{D}_2) = \mathbf{DPCHC} \vee \mathbf{V}(\mathbf{D}_2)$ .

*Proof.* Use Theorem 5.3 and Proposition 5.8.  $\square$

As an immediate consequence of the above theorem, we get a solution of the problem of axiomatization of  $\mathbf{DPCSHC} \vee \mathbf{V}(\mathbf{D}_2)$  mentioned in the introduction. Several more applications of the above theorem will be given in the next two sections.

## 6. SOME APPLICATIONS OF THEOREM 5.9

Here are some more consequences of our main result (Theorem 5.9).

For  $n \in \mathbb{N}$ , let  $\mathbf{C}_n^{\text{dp}}$  denote the  $n$ -element **DPCSH**-chain (= **DPCH**-chain = **DPCStH**-chain) and  $\mathbf{V}(\mathbf{C}_n^{\text{dp}})$  denotes the variety generated by  $\mathbf{C}_n^{\text{dp}}$ . It should be pointed out that  $\mathbf{C}_3^{\text{dp}}$  was denoted by  $\mathbf{L}_1^{\text{dp}}$ , the 3-element **DPCH**-chain, in [24].

**COROLLARY 6.1.** *Let  $\mathcal{L}$  be the lattice of nontrivial subvarieties of  $\mathbf{JIDLN}_1$ . Then*

- (1)  $\mathcal{L}$  is isomorphic to the direct product  $(\omega + 1) \times \mathbf{2}$ , where the chains  $(\omega + 1)$  and  $\mathbf{2}$  are viewed as lattices.
- (2)  $\mathbf{JIDLN}_1$  and  $\mathbf{DPCHC}$  are the only two elements of infinite height in the lattice  $\mathcal{L}$ .
- (3)  $\mathbf{V} \in \mathcal{L}$  is of finite height iff  $\mathbf{V}$  is either  $\mathbf{D}_2$  or  $\mathbf{C}_n^{\text{dp}}$ , for some  $n \in \mathbb{N}$ , or  $\mathbf{C}_m^{\text{dp}} \times \mathbf{D}_2$ , for some  $m \in \mathbb{N}$ .

As mentioned earlier, the axiomatizations of the variety **DPCSHC** relative to **DPCSHC** and all of its subvarieties were given in [24]. The lattice of subvarieties of **DPCSHC** is an  $\omega + 1$ -chain—this fact was implicit in [24, Section 13].

We now give bases to all subvarieties of  $\mathbf{JIDLN}_1$ , relative to  $\mathbf{JIDLN}_1$  in corollaries 6.2-6.5.

**COROLLARY 6.2.** *The variety **DPCSHC** is defined, modulo  $\mathbf{JIDLN}_1$ , by*

$$x \vee x' \approx 1.$$

*Proof.* Observe that  $\mathbf{DPCSHC} \models x \vee x' \approx 1$  and  $\mathbf{V}(\mathbf{D}_2) \not\models x \vee x' \approx 1$ , and then apply Theorem 5.8.  $\square$



**COROLLARY 6.3.** *The variety  $\mathbf{V}(\mathbf{D}_2)$  is defined, modulo  $\mathbf{JIDLN}_1$ , by*  

$$x'' \approx x.$$

**COROLLARY 6.4.** *The variety  $\mathbf{V}(\mathbf{C}_n^{\text{dp}}) \vee \mathbf{V}(\mathbf{D}_2)$  is defined, modulo  $\mathbf{JIDLN}_1$ , by*

$$(C_n) \ x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \rightarrow x_2) \vee (x_2 \rightarrow x_3) \vee \cdots \vee (x_{n-1} \rightarrow x_n) = 1.$$

*Proof.* Let  $\mathbf{C}_n^{\text{dp}}$  be the **DPCSH**-chain such that  $C_n^{\text{dp}} = \{0, a_1, a_2, \dots, a_{n-2}, 1\}$ , where  $0 < a_1 < a_2 < \cdots < a_{n-2} < 1$ . We now prove that  $\mathbf{C}_n^{\text{dp}} \models (C_n)$ . Let  $\langle c_1, c_2, \dots, c_n \rangle \in C_n^{\text{dp}}$  be an arbitrary assignment in  $C_n^{\text{dp}}$  for the variables such that  $c_i$  is the value of  $x_i$ , for  $i = 1, \dots, n$ . If  $c_i \leq c_{i+1}$  for some  $i$ , then  $c_i \rightarrow c_{i+1} = 1$ , as  $\mathbf{L}$  is a Heyting algebra, and hence, the identity holds in  $\mathbf{C}_n^{\text{dp}}$ . So, we assume that  $c_i > c_{i+1}$ , for  $i = 1, 2, \dots, n$ . Then,  $c_1 = 1$  since  $|C_n^{\text{dp}}| = n$ , implying that  $(C_n)$  holds in  $\mathbf{C}_n^{\text{dp}}$ . Next, suppose that  $\mathbf{V}$  is the subvariety of  $\mathbf{JIDLN}_1$  satisfying  $(C_n)$ . Then, by Theorem 2.8,  $\mathbf{V}$  is a discriminator variety. So, let  $\mathbf{L}$  be a simple algebra in  $\mathbf{V}$ . Then, it follows from the Theorem 5.9 (or Theorem 5.8) that the semi-Heyting reduct of  $\mathbf{L}$  is a Heyting chain or  $\mathbf{L} \cong \mathbf{D}_2$ . Suppose that the semi-Heyting reduct of  $\mathbf{L}$  is a Heyting chain, and assume  $|L| > n$ , then there exist  $b_1, b_2, \dots, b_{n-1} \in L$  such that  $0 < b_1 < \cdots < b_{n-1} < 1$ . Since  $\mathbf{L} \models (C_n)$ , we can assign  $\langle b_{n-1}, b_{n-2}, \dots, b_1, 0 \rangle$  for  $\langle x_1, x_2, \dots, x_{n-1}, x_n \rangle$ . Then,  $b_{n-1} \vee (b_{n-1} \rightarrow b_{n-2}) \vee \cdots \vee (b_1 \rightarrow 0) = 1$ , yielding

$$b_{n-1} \vee b_{n-2} \vee \cdots \vee b_1 \vee 0 = 1,$$

implying that  $b_{n-1} = 1$ , which is a contradiction. Thus we have  $|L| \leq n$ , and we conclude that  $\mathbf{V} = \mathbf{V}(\mathbf{C}_n^{\text{dp}})$ . Next, suppose  $\mathbf{L} \cong \mathbf{D}_2$ ; then clearly  $\mathbf{V} \cong \mathbf{V}(\mathbf{D}_2)$ , completing the proof.  $\square$

**COROLLARY 6.5.** *The variety  $\mathbf{V}(\mathbf{C}_n^{\text{dp}})$  is defined, modulo  $\mathbf{JIDLN}_1$ , by*

- (1)  $x \vee x' \approx 1$
- (2)  $x_1 \vee x_2 \vee \cdots \vee x_n \vee (x_1 \rightarrow x_2) \vee (x_2 \rightarrow x_3) \vee \cdots \vee (x_{n-1} \rightarrow x_n) = 1.$

**COROLLARY 6.6.** *The variety  $\mathbf{V}(\mathbf{C}_3^{\text{dp}}) \vee \mathbf{V}(\mathbf{D}_2)$  is defined, modulo  $\mathbf{JIDLN}_1$ , by*

$$x \wedge x'^{*'} \leq y \vee y^* \text{ (Regularity).}$$

*It is also defined, modulo  $\mathbf{JIDLN}_1$ , by*

$$x \wedge x' \leq y \vee y^*.$$

The variety  $\mathbf{V}(\mathbf{L}_1^{\text{dp}})(=\mathbf{V}(\mathbf{C}_3^{\text{dp}}))$  is axiomatized in Corollary 6.5 and [24]. Here is yet another axiomatization for it.

**COROLLARY 6.7.** *The variety  $\mathbf{V}(\mathbf{L}_1^{\text{dp}})$  is defined, modulo  $\mathbf{JIDLN}_1$ , by*

- (1)  $x \wedge x'^{*'} \leq y \vee y^* \text{ (Regularity)}$
- (2)  $x^{*'} = x^{**}.$

## 7. AMALGAMATION PROPERTY

We now examine the Amalgamation Property for subvarieties of the variety  $\mathbf{JIDLN}_1$ . For this purpose, we need the following theorem from [11].

**THEOREM 7.1.** *Let  $\mathbf{K}$  be an equational class of algebras satisfying the Congruence Extension Property (CEP), and let every subalgebra of each subdirectly irreducible algebra in  $\mathbf{K}$  be subdirectly irreducible. Then  $\mathbf{K}$  satisfies the Amalgamation Property if and only if whenever  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  are subdirectly irreducible algebras in  $\mathbf{K}$  with  $\mathbf{A}$  a common subalgebra of  $\mathbf{B}$  and  $\mathbf{C}$ , the amalgam  $(\mathbf{A}; \mathbf{B}, \mathbf{C})$  can be amalgamated in  $\mathbf{K}$ .*

The proof of the following lemma is straightforward. We note that “ $\leq$ ” abbreviates “is a subalgebra of” in the next lemma.

Recall from [24] that the proper, nontrivial subvarieties of  $\mathbf{DPCSHC}$  are precisely the subvarieties of the form  $\mathbf{V}(\mathbf{C}_n^{\text{dp}})$ , for  $n \in \mathbb{N}$ .

**LEMMA 7.2.** *Let  $m, n \in \mathbb{N}$ . Then*

$$\mathbf{C}_m^{\text{dp}} \leq \mathbf{C}_n^{\text{dp}}, \text{ for } m \leq n.$$

**THEOREM 7.3.** *Every subvariety of  $\mathbf{DPCSHC}$  has Amalgamation Property.*

*Proof.* It follows from Theorem 2.8 and Theorem 3.9 that  $\mathbf{JIDLN}_1$  is a discriminator variety and hence has CEP. Also, from Theorem 5.8 we obtain that every subalgebra of each subdirectly irreducible (= simple) algebra in  $\mathbf{DPCSHC}$  is subdirectly irreducible. Let  $\mathbf{V}$  be a subvariety of  $\mathbf{DPCSHC}$ . Then, in view of Theorem 7.1 that we need only consider an amalgam  $(\mathbf{A}; \mathbf{B}, \mathbf{C})$ , where  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  are simple in  $\mathbf{V}$  and  $\mathbf{A}$  a subalgebra of  $\mathbf{B}$  and  $\mathbf{C}$ . First, suppose  $\mathbf{V} = \mathbf{V}(\mathbf{C}_n^{\text{dp}})$  for some  $n$ . Then  $\mathbf{B}$  and  $\mathbf{C}$  are  $\mathbf{DPCSHC}$ -chains. Then, in view of the preceding lemma, it is clear that the amalgam  $(\mathbf{A}; \mathbf{B}, \mathbf{C})$  can be amalgamated in  $\mathbf{V}$ . Next, suppose  $\mathbf{V} = \mathbf{DPCSHC}$ . Then it is clear that the amalgamation can be achieved as in the previous case, thus completing the proof.  $\square$

We conclude this section with the following remark: Since every subvariety  $\mathbf{V}$  of  $\mathbf{DPCSHC}$  has CEP and Amalgamation Property, it follows from Banachewski [5] that  $\mathbf{V}$  has enough injectives.

## 8. Concluding Remarks and Open Problems

We like to remark here that all finite simple algebras considered in this paper are quasiprimal. We now mention some open problems for further research.

**Problem 1:** For each variety  $\mathbf{V}(\mathbf{L})$ , where  $\mathbf{L}$  is a simple algebra in  $\mathbf{JIDLN}_1$ , find a Propositional Calculus  $\mathbf{P}(\mathbf{V})$  such that the equivalent algebraic semantics for  $\mathbf{P}(\mathbf{V})$  is  $\mathbf{V}(\mathbf{L})$  (with 1 as the designated truth value,

using  $\rightarrow$  and  $'$  as implication and negation respectively). (For the variety  $\mathbf{V}(\mathbf{2}^e)$ , the answer is, of course, well known: Classical Propositional Calculus.)

These (many-valued) logics might be of interest in switching circuit theory and in computer science.

**Problem 2:** Describe simples in the variety of pseudocommutative **JIDLN**-algebras, and more generally in the variety **JIDSH<sub>1</sub>**.

Problem 3: Investigate the lattice of subvarieties of **DQDLN<sub>1</sub>**, and more generally of **DQDLN**.

Problem 4: Describe simples in **JIDStSH<sub>1</sub>**.

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